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BANACH SPACES WITH A WEAK COTYPE 2 PROPERTY

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ABSTRACT

We study the Banach spaces X with the following property: there is a number δ in $]0,1[$ such that for some constant C, any finite dimensional subspace $E \subset X$ contains a subspace $F \subset E$ with $\dim F \geqq \delta \dim E$ which is C-isomorphic to a Euclidean space. We show that if this holds for some δ in $]0,1[$ then it also holds for all δ in $]0,1[$ and we estimate the function $C = C(\delta)$. We show that this property holds iff the "volume ratio" of the finite dimensional subspaces of X are uniformly bounded. We also show that (although X can have this property without being of cotype 2) $L_2(X)$ possesses this property iff X if of cotype 2. In the last part of the paper, we study the K-convex spaces which have a dual with the above property and we relate it to a certain extension property.

In [5], it is proved that every Banach space X of cotype 2 enjoys the following property:

For each $\varepsilon > 0$, there is a number $\delta_0 = \delta_0(\varepsilon) > 0$ such that,
every finite dimensional subspace $E \subset X$ contains a subspace
(*) $F \subset E$ of dimension $\dim F \geqq \delta_0 \dim E$ which is
$(1+\varepsilon)$-isomorphic to a Euclidean space.

Conversely, this property implies that X is of cotype q for every $q > 2$. However, the paper [5] also includes an example due to W. B. Johnson showing that the preceding property (*) does *not* imply that X is of cotype 2.

In the present note, we will investigate the above property (*) in more detail. We will give an equivalent formulation which resembles the notion of "cotype 2", from which it will follow easily that, if $p \leqq 2$, $L_p(X)$ has the above property iff X is of cotype 2.

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Furthermore, we will be concerned by the following related question:

Consider a space X with the above property (*) and fix a number δ in $]0,1[$ (with $\delta > \delta_0$). Is it true that every finite dimensional $E \subset X$ contains a subspace $F \subset E$ of dimension $\dim F \geqq \delta \dim E$ which is C_δ-isomorphic to a Euclidean space, where C_δ is a number depending on δ only?

We will answer this question affirmatively (giving an estimate of C_δ when $\delta \to 1$) and we will also show that the above property (*) is equivalent to the following:

$$(**)\quad \begin{cases} \text{There is a constant } A \text{ such that every finite dimensional subspace } E \subset X \\ \text{satisfies } \mathrm{vr}(E) \leqq A, \text{ where } \mathrm{vr}(E) \text{ denotes the "volume ratio"} \\ \text{in the sense of [23], which is defined below.} \end{cases}$$

For a finite dimensional (in short f.d.) space E, let us denote by B_E the unit ball of E and let ξ_E be the maximal volume ellipsoid included in B_E; then the "volume ratio" of E is defined as

$$\mathrm{vr}(E) = \left(\frac{\mathrm{vol}(B_E)}{\mathrm{vol}(\xi_E)}\right)^{1/n}, \qquad \dim E = n.$$

The proof of the implication $(*) \Rightarrow (**)$ is closely related to the recent paper [1]. There, it is proved that every cotype 2 space possesses the preceding property (**). Our proof is different from the one in [1], and yields a somewhat stronger statement even in the case when E is of cotype 2.

Recall that by the results of [10] (or [5]), if a f.d. space E is C-isomorphic to a Euclidean space, then it contains for each $\varepsilon > 0$ a subspace $F \subset E$ which is $(1+\varepsilon)$-isomorphic to a Euclidean space and of dimension $\dim F \geqq \delta'' \dim E$ where $\delta'' > 0$ is a number depending only on C and $\varepsilon > 0$. Therefore, in the above property (*) we may as well take ε fixed (say $\varepsilon = 1$). In the sequel, we will denote by P_S the orthogonal projection onto a subspace S of a Hilbert space.

Let us recall the definition of the Banach–Mazur distance between two spaces E, F which are isomorphic: $d(E,F) = \inf\{\|T\|\,\|T^{-1}\|\}$ where the infimum runs over all isomorphisms $T: E \to F$.

We will almost always consider the distance to a Euclidean space $d(F, l_2^{\dim F})$ and we will use the abreviated notation

$$d_F = d(F, l_2^{\dim F}).$$

We then introduce the following number for a Banach space X and for $0 < \delta < 1$:

$$d_X(\delta) = \sup_{\substack{E \subset X \\ E \text{ f.d.}}} \inf \{ d_F \mid F \subset E, \dim F \geqq \delta \dim E \}.$$

In other words, $d_X(\delta)$ is the smallest constant C such that every f.d. subspace $E \subset X$ contains a subspace $F \subset E$ with $\dim F \geqq \delta \dim E$ such that $d_F \leqq C$. (Note that $d_X(\delta)$ may be infinite.) Our main result is the following theorem.

THEOREM 1. *Let X be a space such that $d_X(\delta_0) < \infty$ for some δ_0 in $]0,1[$. Then $d_X(\delta) < \infty$ for all δ in $]0,1[$. Moreover, we have an estimate of the form*

$$d_X(\delta) \leqq C'(1-\delta)^{-1} |\mathrm{Log}(C'(1-\delta)^{-1})| \qquad \forall \delta \in]0,1[\tag{1}$$

for some constant C' (depending only on δ_0 and $d_X(\delta_0)$). For the constant C', we will obtain the estimate

$$C' \leqq \beta' d_X(\delta_0) \delta_0^{-1}$$

for some numerical constant β'.

We will then deduce from Theorem 1

THEOREM 2. *The following properties of a Banach space X are equivalent.*
(i) $d_X(\delta_0) < \infty$ *for some δ_0 in $]0,1[$.*
(ii) *There is a constant A such that* $\mathrm{vr}(E) \leqq A$ *for all f.d. subspaces E of X.*

REMARK. The preceding theorem, together with known facts about spaces with a bounded volume ratio, has the following consequence:

Let X be a space such that $d_X(\delta_0) < \infty$ for some δ_0 in $]0,1[$. Then every f.d. subspace E of X has a basis $(e_1, \ldots, e_n)$ such that for any δ in $]0,1[$ and *any* $A \subset \{1, \ldots, n\}$ with $|A| \leqq \delta n$ the vectors $\{e_i \mid i \in A\}$ span a subspace F_A satisfying $d_{F_A} \leqq C(\delta)$, where $C(\delta)$ is a constant depending only on δ.

For the proof of Theorem 1, we need to introduce some notations:

For any operator $u : l_2^n \to X$, we define

$$l(u) = \left(\int \| u(\alpha) \|^2 d\gamma_n(\alpha) \right)^{1/2}$$

where γ_n denotes the canonical Gaussian measure on $\mathbf{R}^n$. For any bounded operator $u : l_2 \to X$, we let

$$l(u) = \sup\{ l(uv) \mid v : l_2^n \to l_2, n \in \mathbf{N}, \| v \| \leqq 1 \}.$$

For more details on this definition, cf. e.g. [4]. We recall that the k-th approximation number, denoted by $a_k(u)$, of an operator u between Banach spaces is the distance of u to the set of operators of rank less than k. In the proof of Theorem 1, we will use the next result.

THEOREM 3. *Under the same assumption as in Theorem* 1, *there is a constant* C'' *such that, for every* n *and every* $u: l_2^n \to X$, *there is a subspace* $S \subset l_2^n$ *of codimension less than* k *such that*

$$\| u_{|S} \| \leqq C'' k^{-1/2} l(u).$$

In other words, we have

$$\sup_{k \geqq 1} k^{1/2} a_k(u) \leqq C'' l(u). \tag{2}$$

Moreover, we will obtain the following bound for the constant $C'' \leqq \beta d_X(\delta_0)\delta_0^{-1}$ *for some numerical constant* β.

REMARK. It is well known, cf. [16], that there is a numerical constant B such that, for any operator u,

$$\pi_2(u) \leqq B \sum_{k \geqq 1} k^{-1/2} a_k(u).$$

Hence, if u is of rank n,

$$\pi_2(u) \leqq B' \mathrm{Log}(n+1) \sup_{k \geqq 1} k^{1/2} a_k(u), \tag{3}$$

for some numerical constant B'.

Therefore, it follows from (2) and (3) that, for any n-dimensional subspace $E \subset X$ and any $u: l_2 \to E$, we have

$$\pi_2(u) \leqq C'' B' \mathrm{Log}(n+1) l(u).$$

This means that the Gaussian cotype 2 constant of E is majorized by $C''B'\mathrm{Log}(n+1)$. In particular, we recover the known fact (cf. [5]) that any space satisfying (*) must be of cotype q for all $q > 2$.

Let us recall the following known fact.

LEMMA 4. *Let* F *be a Banach space and let* $u: l_2^k \to F$ *be an operator. Then, for any* m, *there is a subspace* $S \subset l_2^k$ *with* $\dim S > k - m$ *such that*

$$\| u_{|S} \| \leqq m^{-1/2} d_F l(u).$$

PROOF. By an easy inductive argument, there is an orthonormal basis (e_i) of

l_2^k such that $\|ue_i\| \geqq a_i(u)$ $\forall i = 1,\dots,k$. We have then

$$\left(\sum a_i(u)^2\right)^{1/2} \leqq \left(\sum \|ue_i\|^2\right)^{1/2}$$

hence

$$\leqq d_F l(u).$$

Therefore,

$$a_m(u) \leqq m^{-1/2} d_F l(u),$$

which is equivalent to the conclusion of Lemma 4. q.e.d.

Proof of Theorem 3. Consider $u : l_2^n \to X$. Clearly, by an obvious perturbation argument, we may assume that $\ker u = \{0\}$. Let $\alpha = \delta_0/2$ and let $E = u(l_2^n)$. It is obviously no loss of generality to assume that $\alpha = 1/K$ for some integer K and that $n = K^m$ for some integer m. In this way, we avoid all the irrelevant problems of divisibility. By the definition of $d_X(\delta_0)$, there is a subspace $F \subset E$ with $\dim F = \delta_0 n$ and $d_F \leqq d_X(\delta_0)$. Applying Lemma 4 to $u_{|u^{-1}F}$, we find a subspace $S_1 \subset l_2^n$ with $\dim S_1 = \alpha n$ such that

$$\|u_{|S_1}\| \leqq d_X(\delta_0)(\alpha n)^{-1/2} l(u).$$

Note that $\dim S_1^\perp = (1-\alpha)n$. We then repeat the above construction with $S_1^\perp$ in the place of l_2^n.

We find $S_2 \subset S_1^\perp$ with $\dim S_2 = \alpha(1-\alpha)n$ and

$$\|u_{|S_2}\| \leqq d_X(\delta_0) l(u)(\alpha(1-\alpha)n)^{-1/2}.$$

Next, we replace $S_1^\perp$ by $(S_1 \oplus S_2)^\perp$ and repeat the construction. After t steps, we find pairwise orthogonal subspaces $S_1,\dots,S_t$ such that $\Sigma_{i=1}^{i=t} \dim S_i = [1-(1-\alpha)^t]n$ and

$$\|u_{|S_i}\| \leqq d_X(\delta_0) l(u)[\alpha(1-\alpha)^{i-1} n]^{-1/2}.$$

This implies

$$\|u_{|S_1 \oplus \cdots \oplus S_t}\| \leqq \left(\sum_1^t \|u_{|S_i}\|^2\right)^{1/2}$$

$$\leqq \alpha^{-1/2} d_X(\delta_0) l(u) n^{-1/2} \left(\sum_0^{t-1} (1-\alpha)^{-i}\right)^{1/2}$$

$$\leqq d_X(\delta_0)\alpha^{-1} l(u) n^{-1/2}(1-\alpha)^{-(t-1)/2}.$$

Now let k be any integer $\leqq n$. Let $k_t = \operatorname{codim}(S_1 \oplus \cdots \oplus S_t) = (1-\alpha)^t n$. Finally consider the smallest t such that $k_t < k$ and let $S = S_1 \oplus \cdots \oplus S_t$. Then codim $S < k$ and (since $k_{t-1} \geqq k$) $\| u_{|S} \| \leqq d_X(\delta_0)\alpha^{-1} l(u) k^{-1/2}$. This completes the proof of Theorem 3.

COROLLARY 5. *For a Banach space X, the following properties are equivalent*:

(i) $\exists \delta_0 \in]0,1[$ *such that* $d_X(\delta_0) < \infty$.

(ii) $\exists \delta \in]0,1[$, $\exists C < \infty$ *such that*

$$\forall n\ \forall u : l_2^n \to X \qquad a_{[\delta n]}(u) \leqq C n^{-1/2} l(u).$$

(iii) *There is a constant C such that, for all compact operators* $u : l_2 \to X$, *we have*

$$\sup_k k^{1/2} a_k(u) \leqq C l(u).$$

Moreover, these properties imply the following one:

(iv) *There is a constant C such that, for any finite sequence* $(x_i)_{i \leqq n}$ *such that*

$$\forall (\alpha_i) \in \mathbf{R}^n \qquad \sup |\alpha_i| \leqq \left\| \sum \alpha_i x_i \right\|, \tag{4}$$

we have

$$\sqrt{n} \leqq C \left(\int_{\mathbf{R}^n} \left\| \sum_1^n \alpha_i x_i \right\|^2 \gamma_n(d\alpha) \right)^{1/2}. \tag{5}$$

PROOF. The proof that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) is implicit in the proof of Theorem 3. Let us show that (iii) $\Rightarrow$ (i). The proof follows by a well-known argument. Given an n-dimensional subspace $E \subset X$, we know that there is an isomorphism $u : l_2^n \to E$ such that $\| u \| \leqq 1$ and $\pi_2(u^{-1}) \leqq \sqrt{n}$. It follows that for any $S \subset l_2^n$ of dimension $> n - k$ we have

$$\sqrt{n-k} = \pi_2(\mathrm{Id}_S) \leqq \| u_{|S} \| \pi_2(u^{-1}) \leqq \sqrt{n} \| u_{|S} \|$$

hence $\| u_{|S} | \geqq (1 - k/n)^{1/2}$ so that $a_k(u) \geqq (1 - k/n)^{1/2}$. Taking $k = [n/2]$, we deduce from (iii) that $l(u) \geqq C^{-1}[n/2]^{1/2} 2^{-1/2}$. Then (recalling that $\| u \| \leqq 1$), we deduce immediately from the results of [5] that there is a subspace $F \subset E$ with $\dim F \geqq \delta_0 n$ and $d_F \leqq 2$ where $\delta_0 = \beta \cdot C^{-2}$ for some numerical constant β. This proves that (iii) $\Rightarrow$ (i). Finally, let us show that (iii) $\Rightarrow$ (iv). Consider $(x_i)_{i \leqq n}$ in X satisfying (4). Let E be the span of $(x_i)_{i \leqq n}$. Consider $u : l_2^n \to E$ defined by $u(\alpha) = \Sigma_1^n \alpha_i x_i\ \forall \alpha \in \mathbf{R}^n$. Then, clearly u^{-1} satisfies $\pi_2(u^{-1}) \leqq \sqrt{n}$ (indeed $u^{-1} = iv$ where $v : E \to l_\infty^n$ satisfies $\| v \| \leqq 1$ by (4) and $i : l_\infty^n \to l_2^n$ is the identity map).

Now, if we assume (iii), $\exists S \subset l_2^n$ such that

$$\dim S = [n/2] \quad \text{and} \quad \| uP_S \| \leqq C(2n^{-1})^{1/2} l(u).$$

Hence, we have

$$\begin{aligned} [n/2] = \dim S &= \operatorname{tr}(u^{-1}uP_S) \\ &\leqq \pi_2(u^{-1})\pi_2(uP_S) \\ &\leqq n \| uP_S \| \\ &\leqq Cn^{1/2}2^{1/2} l(u). \end{aligned}$$

Finally, we have $l(u) \geqq (4C)^{-1} n^{1/2}$ (at least for n large enough), which establishes that (iii) $\Rightarrow$ (iv).

REMARK. Using the results of [20], it is easy to give the following application of the preceding corollary: Let S be a K-convex subspace of a Banach space X, then, if X possesses the property (*) above, the same is true for the quotient X/S.

COROLLARY 6. *Let $1 \leqq p \leqq 2$. A Banach space X is of cotype 2 iff there is a δ in $]0,1[$ such that $d_{L_p(X)}(\delta) < \infty$.*

PROOF. It is well known and easy to prove that X is of cotype 2 iff $L_p(X)$ is of cotype 2. Moreover, by [5] every cotype 2 space possesses the property (*). Therefore, the "only if" part is already known. Let us prove the "if" part. Assume that $L_p(X)$ satisfies the property (i) in Corollary 5. Then it must satisfy (iv) in the same statement. Let us denote by (r_n) the Rademacher functions.

Let us consider the subspace of $L_p(X)$ formed by all the functions of the form $\Sigma_1^n r_i x_i$ ($n \in \mathbf{N}$, $x_i \in X$). We denote by $\operatorname{Rad}(X)$ the closure of this space in $L_p(X)$.

Note that if $\| x_i \| \geqq 1$ for $i = 1, \ldots, n$, we have

$$\sup |\alpha_i| \leqq \left\| \sum \alpha_i r_i x_i \right\|_{L_p(X)}.$$

Hence, by the property (iv),

$$\begin{aligned} \sqrt{n} &\leqq C \left(\int \left\| \sum \alpha_i r_i x_i \right\|_{L_p(X)}^2 d\gamma_n(\alpha) \right)^{1/2} \\ &\leqq C \left(\int \left\| \sum r_i \alpha_i x_i \right\|_{L_2(X)}^2 d\gamma_n(\alpha) \right)^{1/2}. \end{aligned}$$

By symmetry and homogeneity, this leads to

$$\sqrt{n}\inf_{i\leqq n}\|x_i\| \leqq C\left(\int \left\|\sum \alpha_i x_i\right\|^2 d\gamma_n(\alpha)\right)^{1/2}$$

It is known that this last inequality implies that X is of cotype 2 (cf. e.g. the argument included in the paper [6] p. 2). q.e.d.

REMARK. Note that we only need that Rad(X) satisfies $(*)$ to conclude that X is of cotype 2.

We now come to the proof of Theorem 1. We will use the following result.

PROPOSITION 7. *There is a function* $\psi :]0,1[\to \mathbf{R}_+$ *with the following property:*
Let X *be a Banach space, let* $v : X \to l_2^n$ *be an operator, then, for each* ε *in* $]0,1[$, *there is a subspace* $S \subset X$ *such that* codim $S < \varepsilon n$ *and*

$$\|v_{|S}\| \leqq \psi(\varepsilon) n^{-1/2} l(v^*).$$

Moreover, $\psi(\varepsilon)$ *tends to infinity as* $O(\varepsilon^{-1})$ *when* $\varepsilon \to 0$.

This result was proved in [12] in an essentially equivalent formulation. It is enough for our purposes in the sequel. However, to obtain better estimates, it is worthwhile to note that in [15] an essentially sharp improvement is obtained on the order of growth of the function ψ, namely $\psi(\varepsilon)$ is $O(\varepsilon^{-1/2})$ when $\varepsilon \to 0$. The latter result can then be reformulated as follows: there is a constant C such that, for all operators $u : l_2^n \to X$ and for all k, there is a subspace $S \subset X^*$ of codimension less than k such that $\|u^*_{|S}\| \leqq Ck^{-1/2} l(u)$. This corresponds to an estimate similar to the one in Theorem 3 but with the so-called Kolmogorov numbers of u (see e.g. [16]), instead of the approximation numbers of u.

In the appendix at the end of this paper we include a proof of a slight refinement of Proposition 7.

We now turn to the proof of Theorem 1. Let E be a Banach space. Then for any operator $v : E \to l_2^n$, we define the dual norm to l

$$l^*(v) = \sup\{\mathrm{tr}(vu) \mid u : l_2^n \to E, l(u) \leqq 1\}.$$

We need to recall two facts.

FACT 1. For any n-dimensional space E, there is an isomorphism $u : l_2^n \to E$ such that $l(u) = l^*(u^{-1}) = \sqrt{n}$.

This is a result due to Lewis [8], applied to the l-norm, as done previously in [4].

FACT 2. There is an absolute constant K such that, for all $u : l_2^n \to E$, we have

$$l(u) \leqq l^*(u^*) K \operatorname{Log}(1 + d_E).$$

This follows from the fact that the K-convexity constant (using here Gaussian variables instead of the Rademacher functions) is majorized by $K \operatorname{Log}(1 + d_E)$ for some absolute constant K. For a proof of the latter, we refer the reader to [18] or [14]. Once this has been clarified, Fact 2 is merely a reformulation of the fact that the orthogonal projection onto the span of independent Gaussian random variables has norm less than $K \operatorname{Log}(1 + d_E)$ on the space $L_2(E)$; see [4] for more details.

PROOF OF THEOREM 1. Let X be such that $d_X(\delta_0) < \infty$. We will use the property obtained in Theorem 3. Let δ be any number in $]0,1[$ and let $\varepsilon = 1 - \delta$. Let E be any n-dimensional subspace of X.

Using Fact 1 above, we find an isomorphism $u : l_2^n \to E$ such that $l(u) = l^*(u^{-1}) = \sqrt{n}$. By Theorem 3, there is a subspace $H \subset l_2^n$ such that $\operatorname{codim} H \leqq \frac{1}{2}\varepsilon n$ such that

$$\| u_{|H} \| \leqq C''(2/\varepsilon)^{1/2} n^{-1/2} l(u). \tag{6}$$

We will denote by $|\ \ |$ the Euclidean norm on l_2^n. We now introduce a number $\rho > 0$ (to be specified later) and we equip the space E with a new norm defined simply by

$$\forall x \in E \qquad \| x \|_\rho = \| x \| + \rho \, | u^{-1} x |.$$

Let $E_1 = u(H) \subset E$. By (6), we have

$$\forall x \in E_1 \qquad \rho | u^{-1} x | \leqq \| x \|_\rho \leqq (C''(2/\varepsilon)^{1/2} + \rho) | u^{-1} x |. \tag{7}$$

Let us denote by E_1^ρ the space E_1 equipped with the norm $\|\ \ \|_\rho$. By (7), we have

$$d_{E_1^\rho} \leqq 1 + C''(2/\varepsilon)^{1/2} \rho^{-1}.$$

Let us denote by $j : E_1^\rho \to E_1$ the inclusion map (i.e. the identity operator). Obviously, $\| j \| \leqq 1$, hence by the ideal property

$$l^*(u^{-1} j) \leqq l^*(u^{-1}) = \sqrt{n}.$$

By Fact 2, it follows that

$$l((u^{-1} j)^*) \leqq A_\rho \sqrt{n}$$

with $A_\rho = K \operatorname{Log}(2 + C''(2/\varepsilon)^{1/2} \rho^{-1})$.

We now apply Proposition 7 to the operator $v = u^{-1}j : E_1^\rho \to l_2^n$. This implies that there is a subspace $S \subset E_1^\rho$ with $\operatorname{codim} S \leqq \frac{1}{2}\varepsilon n$ such that

$$\| u^{-1} j_{|S} \| \leqq \psi(\varepsilon/2) A_\rho.$$

This means that

$$\forall x \in S \qquad (\psi(\tfrac{1}{2}\varepsilon) A_\rho)^{-1} | u^{-1}x | \leqq \| x \| + \rho | u^{-1} x |$$

hence

$$(\psi(\tfrac{1}{2}\varepsilon) A_\rho)^{-1} \{1 - \rho A_\rho \psi(\tfrac{1}{2}\varepsilon)\} | u^{-1} x | \leqq \| x \|.$$

We now observe that (δ and hence $\varepsilon = 1 - \delta$ remaining fixed) we have $\rho A_\rho \to 0$ when $\rho \to 0$. Therefore, we can choose $\rho = F(\delta)$ (a function of δ only) such that

$$\rho A_\rho \psi(\varepsilon/2) = \tfrac{1}{2}. \tag{8}$$

We have then

$$\forall x \in S \qquad \rho | u^{-1} x | \leqq \| x \|. \tag{9}$$

In the other direction, since $S \subset E_1^\rho$, we have by (6)

$$\forall x \in S \qquad \| x \| \leqq C''(2/\varepsilon)^{1/2} | u^{-1} x |. \tag{10}$$

Finally, let us consider S as a subspace of E and let us denote by $\tilde{S} \subset E$ the corresponding normed space. By (9) and (10), we have

$$d_{\tilde{S}} \leqq \rho^{-1} C''(2/\varepsilon)^{1/2}. \tag{11}$$

Moreover $\dim \tilde{S} = \dim S \geqq \dim E_1 - \frac{1}{2}\varepsilon n$, hence

$$\dim \tilde{S} \geqq n - \varepsilon n = \delta n.$$

Let us now come back to the function $\rho = F(\delta)$ determined implicitly by (8). We have

$$\rho \psi(\varepsilon/2) K \log(2 + C''(2/\varepsilon)^{1/2} \rho^{-1}) = \tfrac{1}{2}.$$

Let $\rho = t(\varepsilon/2)^{1/2}$. Using the fact [15] that $(\frac{1}{2}\varepsilon)^{1/2}\psi(\frac{1}{2}\varepsilon)$ remains bounded when $\varepsilon \to 0$, we find that $t = (2/\varepsilon)^{1/2} F(\delta)$ satisfies for some constant C_1

$$t C_1 \operatorname{Log}(2 + 2C''/\varepsilon t) \geqq \tfrac{1}{2}$$

and this implies, for some numerical constant $C_2 > 0$,

$$t \geqq C_2 / | \operatorname{Log}(C''/\varepsilon) | \qquad \text{when } \varepsilon \to 0.$$

Finally, substituting in (11), we obtain

$$d_{\hat{S}} \leqq \frac{C_3 C''}{(1-\delta)} \left| \operatorname{Log} \frac{C''}{1-\delta} \right|$$

for some numerical constant C_3, when $\delta \to 1$.

Equivalently, $d_X(\delta) \leqq C_3 C''(1-\delta)^{-1} |\operatorname{Log} C''(1-\delta)^{-1}|$, which completes the proof of Theorem 1.

In the sequel, we will need to recall the notations in use for the so-called entropy numbers and Gelfand numbers of a compact operator $u : X \to Y$ between Banach spaces (cf. e.g. [16]).

For any compact subset $K \subset Y$, we denote by $N(K, \varepsilon)$ the smallest number of open balls of radius ε which cover K.

We then define

$$e_k(u) = \inf\{\varepsilon > 0 \mid N(u(B_X), \varepsilon) \leqq 2^k\}.$$

Moreover, we define

$$c_k(u) = \inf\{\| u_{|F} \| \mid F \subset E, \operatorname{codim} F < k\}.$$

Note that we have obviously

$$c_k(u) \leqq a_k(u).$$

We now pass to the proof of Theorem 2. We will first establish the following result.

THEOREM 8. *Let E be an n-dimensional space. Assume that there is a constant C and $\alpha > 0$ such that, for all $k \leqq n$, there is a subspace $F \subset E$ of codimension less than k such that*

$$d_F \leqq C(n/k)^\alpha.$$

Let $v : E \to l_2^n$ be an operator. We have then

$$e_n(v) \leqq C\rho_\alpha \frac{\pi_2(v)}{\sqrt{n}} \tag{12}$$

where ρ_α is a constant depending only on α.

Consequently,

$$\left(\frac{\operatorname{vol}(v(B_E))}{V_n} \right)^{1/n} \leqq 2C\rho_\alpha \pi_2(v) n^{-1/2} \tag{13}$$

where V_n denotes the volume of the n-dimensional Euclidean ball.

To prove Theorem 8 we will use the following lemma due to Carl [2]. We sketch the proof for the convenience of the reader.

LEMMA 9. *For each $\alpha > 0$, there is a constant λ_α such that every operator $v : E \to F$ between Banach spaces satisfies*:

$$\forall n \qquad \sup_{k \leqq n} k^\alpha e_k(v) \leqq \lambda_\alpha \sup_{k \leqq n} k^\alpha c_k(v). \tag{14}$$

PROOF. Note that we may embed F into an L_∞ space (isometrically) without changing the left-hand side of (14). Now, if $F = L_\infty$ then $c_k(v) = a_k(v)$ (by the extension property of L_∞) so that it is enough to prove (14) with $a_k(v)$ in the place of $c_k(v)$.

Let us assume that $\sup_{k \leqq n} k^\alpha a_k(v) \leqq 1$, and that $n = 2^N$. Then for every $m \leqq N$, there is an operator v_m such that

$$\operatorname{rank}(v_m) < 2^m \quad \text{and} \quad \|v - v_m\| \leqq 2^{-m\alpha}.$$

Let $v_0 = 0$. Then

$$v = \sum_{m=1}^{N} (v_m - v_{m-1}) + v - v_N.$$

Let $\Delta_m = v_m - v_{m-1}$. We have

$$\operatorname{rank}(\Delta_m) < 2^{m+1} \quad \text{and} \quad \|\Delta_m\| \leqq 2^{-(m-1)\alpha} \cdot 2. \tag{15}$$

Let $K = v(B_E)$, let $K_m = \Delta_m(B_E)$, and let $\varepsilon_m > 0$, to be specified later. Since the dimension of K_m is majorized by (15), we find, using a classical estimate (cf. e.g. [5] p. 58),

$$\forall \varepsilon > 0 \qquad N(K_m, \varepsilon \|\Delta_m\|) \leqq (1 + 2\varepsilon^{-1})^{2^{m+1}}.$$

Observe that for $0 < r < 1$, we have

$$\forall \varepsilon > 0 \quad \forall d \in \mathbf{N} \qquad (1 + 2/\varepsilon)^d \leqq \exp(2/\varepsilon)^r d r^{-1}.$$

Hence, since $K \subset \sum_{m=1}^N K_m + (v - v_N)(B_E)$,

$$N\Big(K, \sum_{m=1}^{N} \varepsilon_m \|\Delta_m\| + 2^{-N\alpha}\Big) \leqq \prod_{m=1}^{N} N(K_m, \varepsilon_m \|\Delta_m\|)$$

$$\leqq \exp \sum_{m=1}^{N} 2^{m+1} r^{-1} (2\varepsilon_m^{-1})^r.$$

Now, consider a number $\beta > \alpha$ and r such that $0 < r < 1$ and $r < 1/\beta$. Let λ be any positive number. We take $\varepsilon_m = \lambda 2^{m\beta} 2^{-N\beta}$.

By elementary computations, we find constants ρ' and ρ'' depending only on α, β and r such that

$$N(K, \lambda\rho' 2^{-N\alpha}) \leqq \exp \lambda^{-r}\rho'' 2^N.$$

Choosing $\lambda^{-r} = (\rho'')^{-1}\mathrm{Log}\, 2$, we finally obtain

$$e_n(v) \leqq \lambda_\alpha n^{-\alpha}$$

for some constant λ_α depending only on α (we take for instance $\beta = 2\alpha$ and $r = (2\alpha + 1)^{-1}$).

Note that we also obtain an estimate of the form

$$e_{[\delta n]}(v) \leqq \lambda_\alpha \delta^{-1/r} n^{-\alpha}$$

for $0 < \delta < 1$.

By homogeneity, we have proved that

$$\forall n \geqq 1 \qquad n^\alpha e_n(v) \leqq \lambda_\alpha \sup_{k \leqq n} k^\alpha c_k(v)$$

which is equivalent to (14).

Proof of Theorem 8. We first recall that if w is an operator between two Hilbert spaces, then $\pi_2(w)$ coincides with the Hilbert–Schmidt norm of w, or equivalently

$$\pi_2(w) = \left(\sum_1^\infty a_k(w)^2 \right)^{1/2} \qquad \text{(cf. e.g. [16]).}$$

It follows that for any $w : F \to l_2^n$ we have

$$\left(\sum a_k(w)^2 \right)^{1/2} \leqq d_F \pi_2(w). \tag{16}$$

Now consider $v : E \to l_2^n$ as in Theorem 8. Let $F \subset E$ be a subspace such that $\operatorname{codim} F < k$ and $d_F \leqq C(n/k)^\alpha$. Then, by (16) we have

$$k^{1/2} a_k(v_{|F}) \leqq d_F \pi_2(v_{|F}) \leqq d_F \pi_2(v).$$

Therefore, there is a subspace $F_1 \subset F$ such that $\dim F - \dim F_1 < k$ for which

$$\| v_{|F_1} \| \leqq d_F k^{-1/2} \pi_2(v).$$

This implies

$$c_{2k}(v) \leqq d_F k^{-1/2} \pi_2(v).$$

Therefore

$$\sup_{k \leq n} k^{\alpha+1/2} c_k(v) \leqq C 2^{\alpha+1/2} n^{\alpha} \pi_2(v).$$

By Lemma 9

$$e_n(v) \leqq \lambda_{\alpha+1/2} C 2^{\alpha+1/2} n^{-1/2} \pi_2(v).$$

This concludes the proof of Theorem 8 since the last assertion is immediate: by the definition of $e_n(v)$ we have $\mathrm{vol}(v(B_E)) \leqq 2^n V_n e_n(v)^n$, so that (13) follows from (12).

Remark. In the particular case when X is of cotype 2 the preceding proof simplifies a great deal. Let us streamline the argument. We consider $E \subset X$ and $u : l_2^n \to E$ such that $\|u\| \leqq 1$ and $\pi_2(u^{-1}) \leqq \sqrt{n}$. Let us denote by $C_2(E)$ the (Gaussian) cotype 2 constant of E. We use first an argument similar to the one for Theorem 1; we introduce the norm $\|x\|_\rho = \|x\| + \rho \|u^{-1}x\|$, we let $E = (E, \| \ \|_\rho)$, and we observe that

$$l((u^{-1} : E_\rho \to l_2^n)^*) \leqq K \operatorname{Log}(2 + 1/\rho) C_2(E) n^{1/2}$$

for some numerical constant K. Therefore, by Proposition 7 with the improvement of [15], there is a subspace $S \subset E_\rho$ with $\operatorname{codim} S < k$ such that

$$\|u^{-1} : S \to l_2^n\| \leqq K'(n/k)^{1/2} \operatorname{Log}(1 + 1/\rho) C_2(E).$$

Then, proceeding as in the proof of Theorem 1, we find

$$\begin{aligned} c_k(u^{-1}) &\leqq K''(n/k)^{1/2} C_2(E) \operatorname{Log}[C_2(E)(n/k)^{1/2} + 1] \\ &\leqq K''' C_2(E)[\operatorname{Log}(C_2(E) + 1)](n/k) \end{aligned}$$

hence by Lemma 9

$$e_n(u^{-1}) \leqq K_4 C_2(E) \operatorname{Log}(C_2(E) + 1)$$

and *a fortiori*

$$\mathrm{vr}(E) \leqq 2 K_4 C_2(E) \operatorname{Log}(C_2(E) + 1)$$

for some numerical constant K_4.

It is conceivable that an estimate such as

$$\mathrm{vr}(E) \leqq \text{constant} \cdot C_2(E)(\operatorname{Log} C_2(E) + 1)^{1/2}$$

holds. Note that (because we are using the Gaussian cotype 2 constant)

$$C_2(l_\infty^n) \approx (n/\mathrm{Log}\, n)^{1/2} \quad \text{and} \quad \mathrm{vr}(l_\infty^n) \geqq C\sqrt{n}$$

so that such an estimate would be optimal.

REMARK. In the preceding remark, we proved in passing that for any δ in $]0,1[$ there is a subspace $S \subset E$ with $\dim S \geqq \delta n$ and

$$d_S \leqq K_5 C_2(E)(1-\delta)^{-1/2}\mathrm{Log}[C_2(E)(1-\delta)^{-1/2}+1]$$

(take $k \cong (1-\delta)n$ in the preceding reasoning).

Similar estimates (with a worse dependence of δ) appeared already in [3] and [12]. The above estimate was obtained recently in [15], but our argument has the advantage of avoiding the iteration technique used in all these papers.

REMARK. It is clear from the proof of Theorem 8 that $\pi_2(v)$ can be replaced by

$$\sup_k k^{1/2} \sup_{\substack{w: l_2^n \to E \\ \|w\| \leqq 1}} a_k(vw).$$

The latter quantity appears in the paper of Pietsh on the so-called Weyl numbers; cf. [17].

PROOF OF THEOREM 2. The implication (i) $\Rightarrow$ (ii) in Theorem 2 is now an easy corollary of Theorem 1 and Theorem 8. The implication (ii) $\Rightarrow$ (i) is already known; cf. [23].

It is also possible to adapt the argument of [1] to show that Theorem 1 implies Theorem 2. However, our proof using Theorem 8 seems simpler and gives more flexibility for the choice of the ellipsoid.

In the last part of this paper, we study the duals of the Banach spaces considered in Theorem 1 assuming moreover that they are K-convex, or equivalently (cf. [19]) that they do not contain l_1^n's uniformly. We recall that a Banach space X is called K-convex if the orthogonal projection onto the closed span of the Rademacher functions in $L_2([0,1])$ defines a bounded operator on $L_2([0,1];X)$.

THEOREM 10. *The following assertions are equivalent for a space X.*

(i) *X is K-convex and there is a δ_0 in $]0,1[$ such that $d_{X^*}(\delta_0) < \infty$.*

(ii) *For all δ in $]0,1[$, there is a constant C_δ such that, for any subspace $S \subset X$ and any operator $u: S \to l_2^n$, there is an orthogonal projection $P: l_2^n \to l_2^n$ with rank $P \geqq \delta n$ and an operator $\tilde{u}: X \to l_2^n$ such that $\tilde{u}_{|S} = Pu$ and $\|\tilde{u}\| \leqq C_\delta \|u\|$.*

(iii) *For some* δ *in* $]0,1[$, *the same as* (ii) *holds.*

$$\begin{array}{ccccc} X & \searrow^{\tilde{u}} & & & \\ \cup & & & & \\ S & \xrightarrow{u} & l_2^n & \xrightarrow{P} & l_2^n \end{array}$$

PROOF. (i) $\Rightarrow$ (ii). We will use the following fact:

If a space Y is K-convex there is a constant λ with the following property: for any subspace $M \subset Y$, let $\sigma : Y \to Y/M$ be the quotient map, for any $v : l_2^n \to Y/M$, there is a "lifting" $\tilde{v} : l_2^n \to Y$ such that $\sigma\tilde{v} = v$ and $l(\tilde{v}) \leqq \lambda l(v)$. This follows rather easily from the definition of K-convexity. In fact, the preceding property even holds for Y arbitrary assuming only that M is K-convex; the constant λ will then depend on M. See [20] for details.

Now assume (i) and consider u as in (ii). Consider $u^* : l_2^n \to X^*/S^\perp$.

Obviously we have $l(u^*) \leqq \sqrt{n}\| u^* \| = \sqrt{n}\| u \|$. By the preceding property, there is an operator $(\widetilde{u^*}) : l_2^n \to X^*$ such that (denoting by $\sigma : X^* \to X^*/S^\perp$ the quotient map) we have

$$\sigma(\widetilde{u^*}) = u^* \qquad \text{and} \qquad l(\tilde{u}^*) \leqq \lambda l(u^*) \tag{17}$$
$$\leqq \lambda \sqrt{n}\| u \|.$$

By Theorem 3, there is a constant C such that

$$\operatorname{Sup} k^{1/2} a_k(\widetilde{u^*}) \leqq C l(u^*)$$

hence we find

$$a_k(\widetilde{u^*}) \leqq C\lambda (n/k)^{1/2} \| u \|.$$

Equivalently, there is an orthogonal projection $P : l_2^n \to l_2^n$ with rank $P > n - k$ such that

$$\| (\widetilde{u^*})P \| \leqq C\lambda (n/k)^{1/2} \| u \|. \tag{18}$$

Let then $w = (\widetilde{u^*})^* : X \to l_2^n$.

We have clearly by (17)

$$w_{|S} = u, \qquad \text{hence } Pw_{|S} = Pu$$

and by (18), $\| Pw \| \leqq C\lambda (n/k)^{1/2} \| u \|$.

This implies (ii). (Note that we find $C_\delta \in O((1-\delta)^{-1/2})$).

(ii) $\Rightarrow$ (iii) is trivial.

Let us prove that (iii) $\Rightarrow$ (i). Assume (iii).

Note that, by Dvoretzky's theorem (see [5]), (iii) implies that X contains uniformly complemented l_2^n's and even that X is locally π-euclidean. Hence, by [19] corollary 2.11, X must be K-convex. We now prove that (iii) implies that X^* possesses the second property in Corollary 5. Consider an operator $u : l_2^n \to X^*$. Let δ be as in (iii).

By Proposition 7, there is a subspace $S \subset X$ with codim $S < \delta n/2$ such that (for some constant C)

$$\| u^*_{|S} \| \leqq Cl(u) n^{-1/2}.$$

By (iii), there is a projection $P : l_2^n \to l_2^n$ with rank $P > \delta n$ and an operator $v : X^* \to l_2^n$ such that $v_{|S} = Pu^*_{|S}$ and $\| v \| \leqq C' \| u^*_{|S} \|$ for some constant C'. Returning to u we find

$$(Pu^* - v)_{|S} = 0$$

hence $\operatorname{rank}(Pu^* - v) < \delta n/2$.

Finally, if $T = u^* - v$

$$\operatorname{rank}(T) < \operatorname{rank}(Pu^* - v) + \operatorname{rank}(1 - P) < (1 - \delta/2)n,$$

and $\| u - T^* \| = \| v \| \leqq C'Cl(u)n^{-1/2}$, so that

$$a_k(u) \leqq C'Cl(u)n^{-1/2} \qquad \text{with } k = [(1 - \tfrac{1}{2}\delta)n].$$

This shows by Corollary 5 that X^* satisfies (i) and this concludes the proof of Theorem 10.

Remark 11. In [9], Maurey proved that any space X of type 2 possesses the following property:

$$(+) \quad \begin{cases} \text{There is a constant } C \text{ such that for any subspace } S \subset X \text{ and for} \\ \text{any } n, \text{ any operator } u : S \to l_2^n \text{ admits an extension } \tilde{u} : X \to l_2^n \\ \text{such that } \| \tilde{u} \| \leqq C \| u \|. \end{cases}$$

It is not known whether, conversely, the property (+) implies that X is of type 2. However, the preceding result gives some information in that direction. Indeed, by [19] a space X is of type 2 iff X is K-convex and X^* is of cotype 2. The preceding theorem says that property (iii) [which is a weak form of (+) above] holds iff X is K-convex and X^* satisfies a weak form of cotype 2 as described in Corollary 5. By the remark after Theorem 3, we find that if X satisfies the properties in Theorem 10 (in particular if X satisfies (+)) then there is a constant B'' such that the type 2 constant of any n-dimensional subspace of X is majorized by $B''\operatorname{Log}(n+1)$. Moreover, we have

COROLLARY 12. *Let X be a Banach space. Let $2 \leqq q < \infty$. Then X is of type 2 iff the space $L_q([0,1];X)$ possesses the property $(+)$.*

PROOF. If X is of type 2, so is $L_q(X)$, hence $L_q(X)$ satisfies $(+)$ by Maurey's theorem [9]. Conversely, if $L_q(X)$ satisfies $(+)$ then by Theorem 10 (and an easy "localization" argument), if $1/p + 1/q = 1$, $L_p(X^*)$ satisfies property (i) in Theorem 10. By Corollary 6, this implies that X^* is of cotype 2. Since by property $(+)$ X cannot contain l_1^n's uniformly, it must be K-convex and hence of type 2, by [19].

Note that in Corollary 12 again it is enough to assume that Rad(X) possesses $(+)$ to conclude that X is of type 2.

Appendix

We will give below a different proof of Proposition 7 with a slight refinement.

Let $v : X \to l_2^n$ be an operator; we define

$$S(v) = \sup_{k \geqq 1} \sqrt{k} e_k(v).$$

Note that this quantity is equivalent to $\sup_{\varepsilon > 0} \varepsilon (\operatorname{Log} N(v(B_X), \varepsilon))^{1/2}$.

By a well-known result in the theory of Gaussian processes, we have

$$S(v) \leqq \mu_1 l(v^*)$$

for some absolute constant μ_1.

This result is due to Sudakov [21] and follows from a classical lemma of Slepian. This shows that the next statement improves Proposition 7.

PROPOSITION 7'. *Let $v : X \to l_2^n$ be as above. Then, for each ε in $]0,1[$, there is a subspace $S \subset X$ with* codim $S < \varepsilon n$ *such that*

$$\| v_{|S} \| \leqq \mu_2 \varepsilon^{-1} n^{-1/2} S(v) \tag{19}$$

for some absolute constant μ_2.

The proof is based on the following.

LEMMA. *There are numerical constants $a > 0$, $b > 0$ satisfying the following. Let $\{y_i\}$ be a subset of l_2^n with at most 2^{ak} elements, $k \leqq n$. Then there is an orthogonal projection P on l_2^n with rank $k - 1$ such that*

$$\| P y_i \| \geqq b (k/n)^{1/2} \| y_i \| \qquad \text{for all } i. \tag{20}$$

PROOF. This result is proved (but not stated) in [7] using the fact that the average of $\|Py_i\|$ over all projections P of rank $(k-1)$ is equivalent to $(k/n)^{1/2}\|y_i\|$ and moreover that the deviation of $\|Py_i\|$ from this average is bounded by a suitable exponential estimate.

PROOF OF PROPOSITION 7'. Assume that $S(v)<1$. Let $1 \leqq k \leqq n$, $m = [ak]$ and $N = 2^m$. By definition of $S(v)$ this implies that there are N points $(x_i)_{i \leqq N}$ in B_X such that $\forall x \in B_X$ $\exists i \leqq N$ such that $\|vx - vx_i\| < m^{-1/2}$. Let $y_i = vx_i$. By the preceding lemma, there is a projection P of rank $k-1$ such that (20) holds. Let $S = \operatorname{Ker} Pv$. Note that codim $S \leqq k-1 < k$. Moreover, if x is in $S \cap B_X$, for some $i \leqq N$ we have $\|vx - vx_i\| < m^{-1/2}$. Hence

$$\|vx\| < m^{-1/2} + \|vx_i\|.$$

Hence by (20)

$$\begin{aligned} &< m^{-1/2} + b^{-1}(n/k)^{1/2}\|Pvx_i\| \\ &= m^{-1/2} + b^{-1}(n/k)^{1/2}\|Pv(x - x_i)\| \\ &\leqq m^{-1/2} + b^{-1}(n/k)^{1/2} \cdot m^{-1/2} \end{aligned}$$

so that finally

$$\|v_{|S}\| \leqq b'(n/k) \cdot n^{-1/2} \qquad \text{for some numerical constant } b'.$$

In other words, we have proved (by homogeneity)

$$\operatorname*{Sup}_{1 \leqq k} kc_k(v) \leqq b' n^{1/2} \sup_{k \geqq 1} \sqrt{k} e_k(v),$$

and this clearly is equivalent to (19). q.e.d.

REFERENCES

1. J. Bourgain and V. D. Milman, *On Mahler's conjecture on the volume of a convex symmetric body and its polar*, preprint I.H.E.S., March 1985; cf. also *Sections euclidiennes et volume des corps convexes symétriques*, C. R. Acad. Sci. Paris A **300**, Ser. I (1985), 435–438.

2. B. Carl, *Entropy numbers, s-numbers, and eigenvalue problems*, J. Funct. Anal. **41** (1981), 290–306.

3. S. Dilworth, *The cotype constant and large Euclidean subspaces of normed spaces*, preprint.

4. T. Figiel and N. Tomczak-Jaegerman, *Projections onto Hilbertian subspaces of Banach spaces*, Isr. J. Math. **33** (1979), 155–171.

5. T. Figiel, J. Lindenstrauss and V. D. Milman, *The dimension of almost spherical sections of convex bodies*, Acta Math. **139** (1977), 53–94.

6. R. C. James, *Nonreflexive spaces of type* 2, Isr. J. Math. **30** (1978), 1–13.

7. W. B. Johnson and J. Lindenstrauss, *Extensions of Lipschitz mappings into a Hilbert space*, Proc. Conf. in Honour of S. Kakutani.

8. D. Lewis, *Ellipsoids defined by Banach ideal norms*, Mathematika **26** (1979), 18–29.

9. B. Maurey, *Un théorème de prolongement*, C. R. Acad. Sci. Paris A **279** (1974), 329–332.

10. V. D. Milman, *New proof of a theorem of A. Dvoretzky on sections of convex bodies*, Funct. Anal. Appl. **5** (1971), 28–37.

11. V. D. Milman, *Almost Euclidean quotient spaces of subspaces of finite dimensional normed spaces*, Proc. Am. Math. Soc. **94** (1985), 445–449.

12. V. D. Milman, *Random subspaces of proportional dimensional of finite dimensional normed spaces; approach through the isoperimetric inequality* (Séminaire d'Analyse Fonctionnelle 84/85, Université Paris VII, Paris), *Banach Spaces*, Missouri Conf., Proceedings, 1984, Springer Lecture Notes in Math., **1166**, pp. 106–115.

13. V. D. Milman, *Volume approach and iteration procedures in local theory of normed spaces*, *Banach Spaces*, Missouri Conf., Proceedings, 1984, Springer Lecture Notes in Math., **1166**, pp. 99–105.

14. V. D. Milman and G. Schechtman, *Asymptotic theory of finite dimensional normed spaces*, Springer Lecture Notes, to appear.

15. A. Pajor and N. Tomczak-Jaegerman, *Subspaces of small codimension of finite-dimensional Banach spaces*, to appear.

16. A. Pietsch, *Operator Ideals*, North-Holland, Amsterdam, 1978.

17. A. Pietsch, *Weyl numbers and eigenvalues of operators in Banach spaces*, Math. Ann. **247** (1980), 149–168.

18. G. Pisier, *Un théorème de factorisation pour les opérateurs linéaires entre espaces de Banach*, Ann. Ecole Nat. Sup. **13** (1980), 23–43.

19. G. Pisier, *Holomorphic semi-groups and the geometry of Banach spaces*, Ann. Math. **115** (1982), 375–392.

20. G. Pisier, *Quotients of Banach spaces of cotype q*, Proc. Am. Math. Soc. **85** (1982), 32–36.

21. V. N. Sudakov, *Gaussian random processes and measures of solid angles in Hilbert space*, Soviet Math. Dokl. **12** (1971), 412–415.

22. S. Szarek, *On Kašin almost Euclidean decomposition of* l_1^n, Bull. Acad. Polon. Sci. **26** (1978), 691–694.

23. S. Szarek and N. Tomczak-Jaegerman, *On nearly Euclidean decompositions for some classes of Banach spaces*, Compos. Math. **40** (1980), 367–385.